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The commutativity of quantized first- and higher-order Hamiltonians*

Kaoru Ikeda
Otaru University of Commerce, 3-5-21 Midori, Otaru-shi, Hokkaido 047, Japan

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Abstract. The key point of the Hamiltonian formalism of Toda molecules is the commutativity of the Hamiltonians $\{\operatorname{tr} y^k, \operatorname{tr} y^\ell\} = 0$, where $y \in GL(n)$ and $\{\,,\}$ is a Poisson bracket associated with the classical r-matrix. To quantize the Toda molecule, we have to consider the q-analogue of the above formula. In this paper, we show the commutativity of the quantized first- and higher-order Hamiltonians $[\operatorname{tr}_q X^m, \operatorname{tr}_q X] = 0$, where X is a matrix of quantum group $GL_q(n)$.

1. Introduction

Let us consider the (n) Toda molecule

$$\begin{cases} \partial_0^2 u_1 = 2\mathrm{e}^{2(u_2-u_1)} & \\ \partial_0^2 u_i = 2(\mathrm{e}^{2(u_{i+1}-u_i)} - \mathrm{e}^{2(u_i-u_{i-1})}) & 1 < i < n \\ \partial_0^2 u_n = -2\mathrm{e}^{2(u_n-u_{n-1})}. & \end{cases} \tag{1.1}$$

This equation is a completely integrable system in the sense of classical mechanics. Liouville's theorem (Arnold 1987) asserts that a system with n degrees of freedom (with a $2n$-dimensional phase space) is integrable, if n-independent involutive Hamiltonians exist. It is not trivial that (1.1) is integrable in the Liouville sense. To show this, many methods have been considered, for example, the co-adjoint orbit method (Adler 1979), the construction of the Poisson structure of discrete Lax operators (Kupershmidt 1985), the quantum-group quasi-classical-limit method (the classical r-matrix method) (Ikeda 1991, Kupershmidt 1991) etc. Let $\hat{A}(GL_q(n))$ be the associative algebra generated by x_{ij} over $\mathbb{C}$ ($i \leqslant i, j \leqslant n$). Put

$$R = q\sum_{1\leqslant i\leqslant n} e_{ii}\otimes e_{ii} + \sum_{i\neq j} e_{ii}\otimes e_{jj} + (q-q^{-1})\sum_{1\leqslant j<i\leqslant n} e_{ij}\otimes e_{ji}$$

where e_{ij} is a (i, j)-matrix element. Let X be a matrix such that $X = \sum_{1\leqslant i,j\leqslant n} x_{ij}e_{ij} = (x_{ij})_{n\times n}$ and $X_1 = X\otimes 1$, $X_2 = 1\otimes X$. I_R is the ideal generated by the components of the matrix $RX_1X_2 - X_2X_1R$. In this paper, we consider the algebra $A(GL_q(n)) = \hat{A}(GL_q(n))/I_R$. For more details on the quantum group see Faddeev *et al* (1988) and Takhtajan (1990). The relations which the generators satisfy are $[x_{ij}, x_{k\ell}] = x_{ij}x_{k\ell} - x_{k\ell}x_{ij} = (q^{\theta(j,\ell)} - q^{-\theta(i,k)})x_{i\ell}x_{kj}$, where

$$\theta(i,j) = \begin{cases} 1 & i<j \\ 0 & i=j \\ -1 & i>j. \end{cases}$$

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Let us expand x_{ij} formally with respect to h such that $x_{ij} = y_{ij} + O(h)$ where $q = e^h$. The elements y_{ij} ($1 \leqslant i, j \leqslant n$) are the generators of a commutative coordinate ring of $GL(n)$ which we denote by $A(GL(n))$. We introduce the Poisson structure to $A(GL(n))$ such that $\{y_{ij}, y_{k\ell}\} = ([x_{ij}, x_{k\ell}]/h) \bmod h$. A practical example is $\{y_{ij}, y_{k\ell}\} = (\theta(j,\ell)+\theta(i,k))y_{i\ell}y_{kj}$. Let y be the $n \times n$ matrix $y = (y_{ij})_{n\times n}$. By easy calculation, we see that $\{\operatorname{tr} y^k, \operatorname{tr} y^\ell\} = 0$ ($\operatorname{tr} y^k$ means the trace of y^k). We can regard $\operatorname{tr} y^k$ ($k \geqslant 1$) as involutive Hamiltonians. Unfortunately, by using the result from Waring (1770) concerning fundamental symmetric polynomials and Newton's formula, we see that the algebraic independent Hamiltonians are $\operatorname{tr} y, \operatorname{tr} y^2, \dots, \operatorname{tr} y^{n-1}$, although the degree of freedom of the phase space (the number of generators of $A(GL(n))$) is $n^2 - 1$. To obtain an integrable system, we have to constrain the freedom of $A(GL(n))$ while retaining compatibility with its Poisson structure. Put $z_{ij} = ({}^t yy)_{ij}$. The Poisson bracket is compatible with this coordinate transformation

$$\{z_{ij}, z_{k\ell}\} = (\theta(j,k) + \theta(i,\ell))z_{ik}z_{j\ell} + (\theta(j,\ell) + \theta(i,k))z_{i\ell}z_{jk}.$$

Moreover, the constraint $z_{ij} = 0$, $|i - j| > 1$ is also consistent with the Poisson bracket. Finally, the degree of freedom of $A(GL(n))$ reduces to $2n - 2$. Put $z = (z_{ij})_{n\times n}$. The Hamiltonian equations $\partial_m z = \{\operatorname{tr} z^m, z\}$ include the (n) Toda molecule. This is the quantum-group quasi-classical-limit method for showing the integrability of a Toda molecule. Recently, the quantum integrable system has been studied in the field of mathematical physics (Reyman 1993, Reyman and Semenov–Tian–Shansky 1993, Seminov–Tian–Shansky 1993). To construct the quantum Toda molecule, we think that we may apply the quasi-classical-limit method to $A(GL_q(n))$. The first key point of quantization is the q-analogue of the trace formula $\{\operatorname{tr} y^k, \operatorname{tr} y^\ell\} = 0$. The q-power of X is defined as follows.

Definition. ${}_qX^1 = X$, ${}_qX^{k+1} = X(C * {}_qX^k)$ where $C = (q^{-\theta(i,j)})_{n\times n}$ and $(A_{ij})_{n\times n} * (B_{ij})_{n\times n} = (A_{ij}B_{ij})_{n\times n}$.

We assume the q-analogue of the trace formula to be

$$[\operatorname{tr}_q X^k, \operatorname{tr}_q X^\ell] = 0 \tag{1.2}$$

where $\operatorname{tr}_q X^k$ is the trace of ${}_qX^k$. In Ikeda (1993), we show that these Hamiltonians are essentially finite, i.e. $\operatorname{tr}_q X^m$ ($m \geqslant n$) are expressed by polynomials of $\det_q X, \operatorname{tr}_q X, \dots, \operatorname{tr}_q X^{n-1}$. In this paper, we show the commutativity of the first Hamiltonian $\operatorname{tr}_q X$ and other higher-order Hamiltonians, i.e.

$$[\operatorname{tr}_q X^m, \operatorname{tr}_q X] = 0 \qquad m \geqslant 2. \tag{1.3}$$

Kupershmidt (1992) tries to solve a similar problem. In his paper, he adopts the q-trace of an ordinary power of X (in this paper, our Hamiltonians are the ordinary trace of the q-power of X). He concluded that the first and second Hamiltonians do not commute with each other for $X \in GL_q(n)$ ($n \geqslant 3$).

We mention the strategy for proving $[\operatorname{tr}_q X^m, \operatorname{tr}_q X] = 0$ briefly. We show this by induction with respect to matrix size n. Because of the result of the previous letter (Ikeda 1993), we may show that $[\operatorname{tr}_q X^n, \operatorname{tr}_q X] = 0$, $X \in GL_q(n+1)$. We introduce the 'order' with respect to indices of generators to $A(GL_q(n+1))$. We show that we can prove that the highest-order part of $[\operatorname{tr}_q X^n, \operatorname{tr}_q X]$ vanishes (we write this $[\operatorname{tr}_q X^n, \operatorname{tr}_q X]_{1,\dots,n+1}$). For monomial $x_{i_1i_2}\dots x_{i_li_{l+1}}$, we define its annihilator as $-$(the product of $x_{i_1i_2}, \dots, x_{i_li_{l+1}}$ with arbitrary order). From the assumption of induction $[\operatorname{tr}_q X^{n-1}, \operatorname{tr}_q X] = 0$, $X \in$

$GL_q(n)$, we see that $[\mathrm{tr}_q X^{n-1}, \mathrm{tr}_q X]_{i,\dots,n}$ is represented by the pairwise summation $(q-q^{-1})\sum$*(monomial + its annihilator)*. Normalizing the order of monomial to its annihilator, we have $(q-q^{-1})^2\sum$(monomial+its annihilator). Repeating this manipulation, $[\mathrm{tr}_q X^{n-1}, \mathrm{tr}_q X]_{1,\dots,n}$ is always expressed as $(q-q^{-1})^\ell\sum$(monomial + its annihilator). We apply this fact to the case of $A(GL_q(n+1))$. Furthermore, we introduce 'class' to the monomials of $A(GL_q(n))$ and let B be this class. We put $v = x_{i_1i_2}\dots x_{\mu\nu}x_{\rho\eta}\dots x_{i_ti_{t+1}}$ as a monomial of $A(GL_q(n))$. If $\mu < (>)\rho$ and $\nu < (>)\eta$, we have

$$v = x_{i_1i_2}\dots x_{\rho\eta}x_{\mu\nu}\dots x_{i_ti_{t+1}} + (-)(q-q^{-1})x_{i_1i_2}\dots x_{\mu\eta}x_{\rho\nu}\dots x_{i_ti_{t+1}}.$$

We use a property such as $B(v) > B(x_{i_1i_2}\dots x_{\mu\eta}x_{\rho\nu}\dots x_{i_ti_{t+1}})$. The simplicity of the first Hamiltonian $\mathrm{tr}_q X$ is available as proof of the commutativity. If $\ell \neq 1$ in (1.2), the calculation is too difficult to prove commutativity. Semenov–Tian–Shansky (1993) have studied the quantum open Toda lattice. Their method involves the quantization of the Kostant–Adler scheme which is based on the linear Poisson bracket. The quantum group is based on the quadratic Poisson bracket. In this paper, we confine our interest to the Hamiltonian structure on the quantum group of $A(GL_q(n))$. At the beginning of the quantum inverse-scattering method, the quantum non-linear Schrödinger equation is considered (Sklyanin 1982). Its 2×2 monodromy matrix

$$T(\lambda) = \begin{pmatrix} A(\lambda) & HB^+(\lambda) \\ B(\lambda) & A^+(\lambda) \end{pmatrix}$$

satisfies the relation $R_0(\lambda-\mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_0(\lambda-\mu)$ where $R_0(\lambda)$ is a certain R-matrix with spectral parameter. It is shown that $\log A(\lambda)$ is a generating function of the local integral of motion of the quantum nonlinear Schrödinger equation. The various quantum integrable models including the quantum nonlinear Schrödinger equation are the origin of the quantum group. We think that to construct the commutative subalgebra, that is the family of quantum Hamiltonians of quantum group $A(GL_q(n))$, by purely algebraic methods, indicates some direction for studying quantum integrable systems. Furthermore, we should study the physical meaning of the definition of the q-power of X.

2. The commutativity of quantized first- and higher-order Hamiltonians

First, we cite the following theorem.

Theorem 1 (Ikeda 1993). For $X \in GL_q(n)$, $\mathrm{tr}_q X^m$ can be represented by a polynomial of $\mathrm{tr}_q X, \mathrm{tr}_q X^2, \dots, \mathrm{tr}_q X^{n-1}$ and $\det_q X$ where $m \geqslant n$.

Sketch of proof. We refer the reader to Ikeda (1993) for a rigorous proof. The matrix X satisfies the q-analogue of the Cayley–Hamilton formula (Zang 1992)

$$_qX^n - {_qX^{n-1}}d^1 + \dots + (-)^{n-1}_q Xd^{n-1} + (-)^n I \det_q X = 0 \tag{2.1}$$

where $d^k = \sum_{i_1<i_2<\dots<i_k} \det_q X_{i_1\dots i_k}$, $X_{i_1\dots i_k}$ is an $i_1,\dots,i_k$ principal minor of X, $\det_q X_{i_1\dots i_k} = \sum_{\sigma\in S_k}(-q)^{\ell(\sigma)}(X_{i_1\dots i_k})_{1\sigma(1)}\dots(X_{i_1\dots i_k})_{k\sigma(k)}$ and $\ell(\sigma)$ is the number of inversions involved in σ. From (2.1), we see that

$$_qX^{n+m} = {_qX^{n+m-1}}d^1 - \dots - (-)^{n-1}_q X^{m+1}d^{n-1} - (-)^n_q X^m \det_q X. \qquad \square$$

From this we may show the following lemma.

Lemma 2. The quantities d^k $(1 \leqslant k \leqslant n-1)$ can be represented by the polynomial of $\mathrm{tr}_q X, \ldots, \mathrm{tr}_q X^{n-1}$.

Proof. We show this lemma by the induction of n. We can trivially verify this lemma for the case $n = 2$ and can assume that $X_{i_1 \ldots i_k}$ satisfies (2.1). Thus, we have

$$(-)^k k \det_q X_{i_1 \ldots i_k} = -\mathrm{tr}_q X^k_{i_1 \ldots i_k} + \mathrm{tr}_q X^{k-1}_{i_1 \ldots i_k} d^1_{i_1 \ldots i_k} - \cdots - (-)^{k-1} \mathrm{tr}_q X_{i_1 \ldots i_k} d^{k-1}_{i_1 \ldots i_k}$$

where $d^j_{i_1 \ldots i_k}$ is the summation of all the jth principal minor q-determinants of $X_{i_1 \ldots i_k}$. Assuming induction gives

$$\det_q X_{i_1 \ldots i_k} = F_k(\mathrm{tr}_q X_{i_1 \ldots i_{k-1}}, \ldots, \mathrm{tr}_q X^k_{i_1 \ldots i_k}). \tag{2.2}$$

Note that because of the algebraic isomorphism between the algebra generated by $x_{i_\mu i_\nu}$ $(1 \leqslant \mu, \nu \leqslant k)$ and $x_{j_\rho j_\eta}$, $(1 \leqslant \rho, \eta \leqslant k)$, the polynomial function F_k does not depend on the choice of $i_1 < \cdots < i_k$. Then, we see $d^k = F_k(\mathrm{tr}_q X, \ldots, \mathrm{tr}_q X^k)$ $(1 \leqslant k \leqslant n-1)$. $\square$

Definition. Put $f = \sum_{i_1, \ldots, i_{k+1}} a_{i_1 \ldots i_k} x_{i_1 i_2} \ldots x_{i_k i_{k+1}} \in A(GL_q(n))$. For m integers $1 \leqslant j_1 < \ldots < j_m \leqslant n$, we define $f_{j_1, \ldots, j_m}$ such that $f_{j_1, \ldots, j_m} = \sum_{\{(i_1, \ldots, i_k)\} = \{j_1, \ldots, j_m\}} a_{i_1 \ldots i_k} x_{i_1 i_2} \ldots x_{i_k i_{k+1}}$ where $\{(i_1, \ldots, i_k)\}$ is a set of numbers which appear in $\{i_1, \ldots, i_k\}$. For example, for $f = x_{12}x_{21} + x_{13}x_{11}^2 x_{22} + 2x_{11}^3 x_{21} + x_{11}^5$, $f_{1,2}$ is equal to $x_{12}x_{21} + 2x_{11}^3 x_{21}$.

Proposition 3. Let f be an element of $A(GL_q(n))$. $f = 0$ iff $f_{j_1, \ldots, j_m} = 0$ for any indices $1 \leqslant j_1 < \ldots < j_m \leqslant n$.

Proof. We will construct the standard form of the polynomial from which we can conclude whether the polynomial is 0 or not. Put $C = \{(i, j) | 1 \leqslant i, j \leqslant n\}$. We introduce order $\prec$ such that $(i, j) \prec (k, \ell) \iff i < k$ or $i = k$ and $j < \ell$. Let D be a set such that $D = \{(i_1, i_2), \ldots, (i_k, i_k + 1) | k \in \mathbb{N}, (i_s, i_{s+1}) \in C\}$. We extend the order $\prec$ to D such that $\{(i_1, i_2), \ldots, (i_k, i_{k+1})\} \prec \{(j_1, j_2), \ldots, (j_m, j_{m+1})\} \iff k < m$ or $k = m$ and $(i_1, i_2) \prec (j_1, j_2)$ or $k = m, (i_1, i_2) = (j_1, j_2)$ and $(i_3, i_4) \prec (j_3, j_4)$ or $, \ldots,$ or $k = m, (i_1, i_2) = (j_1, j_2), \ldots, (i_{k-1}, i_k) = (j_{k-1}, j_k)$ and $(i_k, i_{k+1}) \prec (j_k, j_{k+1})$.

For monomial $v = x_{i_1 i_2} \ldots x_{i_k i_{k+1}}$, we define the integer $B(v)$ such that

$$B(v) = \sharp\{(i_s, i_{s+1}), (i_t, i_{t+1}) | i_s < (>) i_t, i_{s+1} < (>) i_{t+1}\}.$$

If $i < (>) k$ and $j < (>) \ell$, we say x_{ij} is *bad* to $x_{k\ell}$. Moreover, we say the case $i = k$ and $j < (>) \ell$ or $i < (>) k$ and $j = \ell$ as x_{ij} is *neutral* to $x_{k\ell}$ and the case $i < (>) k$ and $j > (<) \ell$ as x_{ij} is *good* to $x_{k\ell}$, respectively (Kupershmidt 1991). For $x_{i_1 i_2} \ldots x_{\mu\nu} x_{\rho\eta} \ldots x_{i_k i_{k+1}}$, exchanging the product order of $x_{\mu\nu}, x_{\rho\eta}$ results in the following expression

$$x_{i_1 i_2} \ldots x_{\rho\eta} x_{\mu\nu} \ldots x_{i_k i_{k+1}} + (q^{\theta(\nu,\eta)} - q^{-\theta(\mu,\rho)}) x_{i_1 i_2} \ldots x_{\mu\eta} x_{\rho\nu} \ldots x_{i_k i_{k+1}}. \qquad \square$$

Let us show the following lemma.

Lemma 4. Put $v = x_{i_1 i_2} \ldots x_{\mu\nu} x_{\rho\eta} \ldots x_{i_k i_{k+1}}$. If $x_{\mu\nu}$ is bad to $x_{\rho\eta}$, we have

$$B(v) > B(x_{i_1 i_2} \ldots x_{\mu\eta} x_{\rho\nu} \ldots x_{i_k i_{k+1}}).$$

Proof. To read the latter half of this paper, we only have to show the case where $x_{i_s i_{s+1}}$ is *good* or *bad* to $x_{\mu\nu}$ and $x_{\rho\eta}$. We can assume that $\mu < \rho$ and $\nu < \eta$. If $x_{i_s i_{s+1}}$ is *good* to both $x_{\mu\nu}$ and $x_{\rho\eta}$, $x_{i_s i_{s+1}}$ is *good* to both $x_{\mu\eta}$ and $x_{\rho\nu}$ (see figure 1). If $x_{i_s i_{s+1}}$ is *bad* to $x_{\mu\nu}$ and *good* to $x_{\rho\eta}$, the following two cases can be considered: (i) $\mu < i_s < \rho, i_{s+1} > \eta$; or (ii) $\rho < i_s, \nu < i_{s+1} < \eta$. In (i), $x_{i_s i_{s+1}}$ is *good* to $x_{\rho\nu}$ and in (ii), $x_{i_s i_{s+1}}$ is *good* to $x_{\mu\eta}$. If $x_{i_s i_{s+1}}$ is *good* to $x_{\mu\nu}$ and *bad* to $x_{\rho\eta}$, the following two cases are considered: (iii) $i_s < \mu, \nu < i_{s+1} < \eta$; or (iv) $\mu < i_s < \rho, i_{s+1} < \nu$. In (iii), $x_{i_s i_{s+1}}$ is *good* to $x_{\rho\nu}$ and in (iv), $x_{i_s i_{s+1}}$ is *good* to $x_{\mu\eta}$ (see figure 2). □

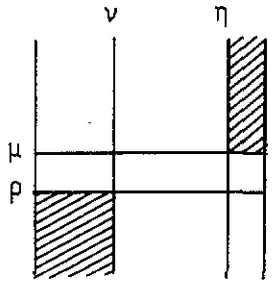

Figure 1. The range of x_{ij}'s which are *good* to both $x_{\mu\nu}$ and $x_{\rho\eta}$.

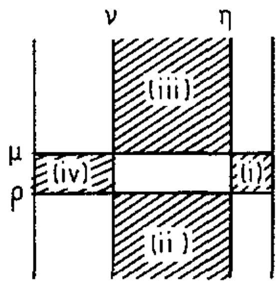

Figure 2. The range of x_{ij}'s which are *good* (*bad*) to $x_{\rho\eta}$ and *bad* (*good*) to $x_{\mu\nu}$ (i) and (ii) ((iii) and (iv)).

Let M be $\max B(x_{j_1 j_2} \dots x_{j_t j_{t+1}})$, where $x_{j_1 j_2} \dots x_{j_t j_{t+1}}$ is a monomial of f. We say a monomial v belongs to the *class* L if $B(v) = L$. Suppose $u = a_{j_1 \dots j_m} x_{j_1 j_2} \dots x_{j_{m-1} j_m}$ is the monomial of f which belongs to class M and $((j_1, j_2), \dots, (j_{m-1}, j_m))$ is the maximum in the following set of indices of the monomial of f: $Y = \{((k_1, k_2), \dots, (k_{s-1}, k_s)) | a_{k_1 \dots k_s} x_{k1k_2} \dots x_{k_{s-1}k_s} \in \text{class } M\}$ with respect to $\prec$. Notice that all the monomials which are products of $x_{j_1 j_2}, \dots, x_{j_{m-1} j_m}$ belong to class M. We normalize the product order of these monomials to that of u and obtain $(a_{j_1 \dots j_m} + \dots) x_{j_1 j_2} \dots x_{j_{m-1} j_m} + \dots$. The new monomials which are created in this process belong to a lower class than class M from lemma 4. If $(a_{j_1 \dots j_m} + \dots) = 0$, we do the same operation with the monomial whose index set is a maximum in $Y - ((j_1, j_2), \dots, (j_{m-1}, j_m))$. If all monomials of f of class M are exhausted by this operation, we repeat the same operation on monomials of f of class $M - 1$. Suppose $f \neq 0$, these operations stop at the form such that

$$f = (a_{\ell_1 \dots \ell_p} + \dots) x_{\ell_1 \ell_2} \dots x_{\ell_{p-1} \ell_p} + \text{others}$$

where $(a_{\ell_1 \dots \ell_p} + \cdots)$ is not 0. From lemma 4, we see that the monomial of *others* cannot yield $x_{\ell_1 \ell_2} \dots x_{\ell_{p-1} \ell_p}$ and, therefore, $f_{\ell_1 \dots \ell_p}$ is not 0. The reverse is clear. □

Theorem 5.

$$[\mathrm{tr}_q X^m, \mathrm{tr}_q X] = 0 \qquad m \geqslant 2 \qquad X \in GL_q(n).$$

Proof. We show this theorem by induction. By easy calculation, we see that $[\mathrm{tr}_q X^2, \mathrm{tr}_q X] = 0$ for $X \in GL_q(3)$ (Ikeda 1991). Thus, theorem 5 is true for $X \in GL_q(3)$ from theorem 1. We assume that $[\mathrm{tr}_q X^m, \mathrm{tr}_q X] = 0$ ($m \geqslant 2$), for $X \in GL_q(k)$ ($k \leqslant n$). Let X be an element of $GL_q(n+1)$. For $\ell < n$ and $k > \ell + 1$, we see that $[\mathrm{tr}_q X^\ell, \mathrm{tr}_q X]_{i_1,\ldots,i_k} = 0$. Furthermore, for $k \leqslant \ell+1$, we have $[\mathrm{tr}_q X^\ell, \mathrm{tr}_q X]_{i_1,\ldots,i_k} = [\mathrm{tr}_q (X_{i_1\ldots i_k})^\ell, \mathrm{tr}_q (X_{i_1\ldots i_k})]_{i_1,\ldots,i_k} = 0$. Similarly, for $k \leqslant n$, we have $[\mathrm{tr}_q X^n, \mathrm{tr}_q X]_{i_1,\ldots,i_k} = 0$. Therefore, we only have to show that $[\mathrm{tr}_q X^n, \mathrm{tr}_q X]_{1,\ldots,n+1}$ is equal to 0. For $X \in GL_q(n)$, put $[\mathrm{tr}_q X^{n-1}, \mathrm{tr}_q X]_{1,\ldots,n} = q^{n-2}A^n_{n-2} + \cdot + q^{-n+2}A^n_{-n+2}$. Since $A^3_1 = A^3_{-1} = 0$, where $[\mathrm{tr}_q X^2, \mathrm{tr}_q X]_{123} = qA^3_1 + q^{-1}A^3_{-1}$ for $X \in GL_q(3)$ (see example), we can assume A^n_{n-2k} is equal to 0 for $k = 1,\ldots,n-1$. We define the annihilator of monomial $x_{i_1i_2}\ldots x_{i_ki_{k+1}}$ as $-$(permutation of $x_{i_1i_2},\ldots,x_{i_ki_{k+1}}$). For example, the annihilators of $x_{12}x_{23}x_{31}$ are $-x_{23}x_{31}x_{12}$, $-x_{31}x_{12}x_{23}$, $-x_{12}x_{31}x_{23}$, $-x_{23}x_{12}x_{31}$, $-x_{31}x_{23}x_{12}$ and $-x_{12}x_{23}x_{31}$. Since A^n_{n-2k} is equal to 0 and

$$
\begin{aligned}
A^n_{n-2k} = & \sum_{\{i_2,\ldots,i_{n-1}\}=\{1,\ldots,k-1,k+2,\ldots,n\}} \\
& \times \sum_{s=1}^{n-1}(q^{\theta(i_{s+1},k+1)} - q^{-\theta(i_s,k+1)})x_{ki_2}\ldots x_{i_sk+1}x_{k+1i_{s+1}}\ldots x_{i_{n-1}k} + \cdots \\
& + \sum_{\{i_2,\ldots,i_{n-1}\}=\{1,\ldots,k-1,k+1,\ldots,n-1\}}\sum_{s=1}^{n-1}(q^{\theta(i_{s+1},n)} - q^{-\theta(i_s,n)})x_{ki_2}\ldots x_{i_sn}x_{ni_{s+1}}\ldots x_{i_{n-1}k} \\
& + \sum_{\{i_2,\ldots,i_{n-1}\}=\{2,\ldots,k-1,k+1,\ldots,n\}} \\
& \times \sum_{s=1}^{n-1}(q^{\theta(i_{s+1},1)} - q^{-\theta(i_s,1)})x_{k+1i_2}\ldots x_{i_s1}x_{1i_{s+1}}\ldots x_{i_{n-1}k+1} + \cdots \\
& + \sum_{\{i_2,\ldots,i_{n-1}\}=\{1,\ldots,k-1,k+2,\ldots,n\}}\sum_{s=1}^{n-1}(q^{\theta(i_{s+1},k)} - q^{-\theta(i_s,k)})x_{k+1i_2}\ldots x_{i_sk}x_{ki_{s+1}}\ldots x_{i_{n-1}k+1}
\end{aligned}
$$

we see that A^n_{n-2k} can be written in the following form:

$$
\begin{aligned}
A^n_{n-2k} = {} & (q-q^{-1}) \sum_{\{i_2\ldots i_n\}=\{1\ldots k-1,k+1\ldots n\},\{j_2\ldots j_n\}=\{1\ldots k,k+2\ldots n\}} (x_{ki_2}\ldots x_{i_nk} - x_{k+1j_2}\ldots x_{j_nk+1}) \\
& + (q-q^{-1}) \sum_{\{i_2\ldots i_n\}=\{1\ldots k,k+2\ldots n\},\{j_2\ldots j_n\}=\{1\ldots k-1,k+1\ldots n\}} \\
& \times (x_{k+1i_2}\ldots x_{i_nk+1} - x_{kj_2}\ldots x_{j_nk})
\end{aligned}
$$

where $-x_{k+1j_2}\ldots x_{j_nk+1}$ and $-x_{kj_2}\ldots x_{j_nk}$ are annihilators of $x_{ki_2}\ldots x_{i_nk}$ and $x_{k+1i_2}\ldots x_{i_nk+1}$, respectively. We normalize the order of products $x_{ki_2}\ldots x_{i_nk}$ and $x_{k+1i_2}\ldots x_{i_nk+1}$ to their annihilators and obtain

$$A^n_{n-2k} = (q-q^{-1})^2 \sum (\pm\text{monomial}).$$

Note that the coefficients of monomials produced by the normalization process are $\pm(q-q^{-1})$ or 0. Since A^n_{n-2k} is equal to 0, we have the following form:

$$A^n_{n-2k} = (q-q^{-1})^2 \sum \{\text{monomial} + (\text{its annihilator})\}.$$

This notation means that the sum of $\pm$monomial can be expressed by the sum of the pairs of monomial + its annihilator. Since A^n_{n-2k} is equal to 0, even if one repeats the same manipulation $\ell - 1$ times, we always have the following form:

$$A^n_{n-2k} = (q-q^{-1})^\ell \sum \{\text{monomial} + (\text{its annihilator})\}. \tag{2.3}$$

For $X \in GL_q(n+1)$, we put

$$[\mathrm{tr}_q X^n, \mathrm{tr}_q X]_{1,\dots,n+1} = q^{n-1} A^{n+1}_{n-1} + \cdots + q^{-n+1} A^{n+1}_{-n+1}.$$

It is easy to see that A^{n+1}_{n+1-2k} has the following form:

$$\begin{aligned} A^{n+1}_{n+1-2k} = (q-q^{-1}) & \sum_{\{i_2,\dots,i_{n+1}\}=\{1,\dots,k-1,k+1,\dots,n+1\},\{j_2,\dots,j_{n+1}\}=\{1,\dots,k,k+2,\dots,n+1\}} \\ & \times (x_{ki_2}\dots x_{i_{n+1}k} - x_{k+1j_2}\dots x_{j_{n+1}k+1}) \\ & + (q-q^{-1}) \sum_{\{i_2,\dots,i_{n+1}\}=\{1,\dots,k,k+2,\dots,n+1\},\{j_2,\dots,j_{n+1}\}=\{1,\dots,k-1,k+1,\dots,n+1\}} \\ & \times (x_{k+1i_2}\dots x_{i_{n+1}k+1} - x_{kj_2}\dots x_{j_{n+1}k}) \end{aligned}$$

where $x_{k+1j_2}\dots x_{j_{n+1}k+1}$ and $x_{kj_2}\dots x_{j_{n+1}k}$ are the annihilators of $x_{ki_2}\dots x_{i_{n+1}k}$ and $x_{k+1i_2}\dots x_{i_{n+1}k+1}$, respectively. To normalize the order of $x_{ki_2}\dots x_{i_{n+1}k}$ and $x_{k+1i_2}\dots x_{i_{n+1}k+1}$ to the order of their annihilators, we must exchange two factors of the monomial such that

$$x_{ki_2}\dots x_{\mu\nu}x_{\rho\eta}\dots x_{i_nk} = x_{ki_2}\dots x_{\rho\eta}x_{\mu\nu}\dots x_{i_nk} + (q^{\theta(\nu,\eta)} - q^{-\theta(\mu,\rho)})x_{ki_2}\dots x_{\mu\eta}x_{\rho\nu}\dots x_{i_nk}.$$

We call $(q^{\theta(\nu,\eta)} - q^{-\theta(\mu,\rho)})x_{ki_2}\dots x_{\mu\eta}x_{\rho\nu}\dots x_{i_nk}$ the new monomial. The coefficient of the new monomials, i.e. $(q^{\theta(\nu,\eta)} - q^{-\theta(\mu,\rho)})$, depends only on the nearest-neighbour indices μ, ν, ρ and η. We can obtain the order of the new monomial from $x_{ki_2}\dots x_{\mu\nu}x_{\rho\eta}\dots x_{i_nk}$ by exchanging indices ν and η. Therefore, we see that there is no difference in the process for generating the new monomial between the two cases A^n_{n-2k} and A^{n+1}_{n+1-2k} and can obtain the form

$$A^{n+1}_{n+1-2k} = (q-q^{-1})^2 \sum \{\text{monomial} + (\text{its annihilator})\}.$$

The process of normalizing the order of product of monomial to its annhilator itself yields new monomials. There is no difference in rules and symmetry which the process of yielding new monomials should obey between n and $n+1$ (see above). Thus, A^{n+1}_{n+1-2k} inherits the property (2.3)

$$A^{n+1}_{n+1-2k} = (q-q^{-1})^{\ell'} \sum \{\text{monomial} + (\text{its annihilator})\}. \tag{2.4}$$

This process is continued until the value of B for any monomial appearing in the summation in (2.4) is equal to 0. In this case, we see that monomial + (its annihilator) = 0. Then, we have A^{n+1}_{n+1-2k} equal to 0. □

Example. For $X \in GL_q(3)$, $[\mathrm{tr}_q X^2, \mathrm{tr}_q X]_{123} = qA^3_1 + q^{-1}A^3_{-1}$

$$\begin{aligned} A^3_1 &= (q-q^{-1})\{(x_{13}x_{32}x_{21} - x_{21}x_{13}x_{32}) + (x_{12}x_{23}x_{31} - x_{23}x_{31}x_{12}) \\ &= (q-q^{-1})^2(x_{13}x_{22}x_{31} - x_{13}x_{31}x_{22}) = 0 \end{aligned}$$

$$\begin{aligned} A^3_{-1} &= (q-q^{-1})\{(x_{23}x_{31}x_{12} - x_{31}x_{12}x_{23}) + (x_{21}x_{13}x_{32} - x_{32}x_{21}x_{13})\} \\ &= (q-q^{-1})^2(x_{22}x_{31}x_{13} - x_{31}x_{22}x_{13}) = 0. \end{aligned}$$

For $X \in GL_q(4)$, $[\mathrm{tr}_q X^3, \mathrm{tr}_q X]_{1234} = q^2 A_2^4 + A_0^4 + q^{-2} A_{-2}^4$

$$\begin{aligned}
A_2^4 = (q-q^{-1})\{&(x_{14}x_{42}x_{23}x_{31} - x_{23}x_{31}x_{14}x_{42}) + (x_{12}x_{24}x_{43}x_{31} - x_{24}x_{43}x_{31}x_{12}) \\
&+ (x_{12}x_{23}x_{34}x_{41} - x_{23}x_{34}x_{41}x_{12}) + (x_{14}x_{43}x_{32}x_{21} - x_{21}x_{14}x_{43}x_{32}) \\
&+ (x_{13}x_{34}x_{42}x_{21} - x_{21}x_{13}x_{34}x_{42}) + (x_{13}x_{32}x_{24}x_{41} - x_{24}x_{41}x_{13}x_{32})\} \\
= (q-q^{-1})^2\{&(x_{24}x_{13}x_{42}x_{31} - x_{13}x_{31}x_{24}x_{42}) + (x_{14}x_{22}x_{43}x_{31} - x_{14}x_{43}x_{31}x_{22}) \\
&+ (x_{23}x_{14}x_{32}x_{41} - x_{14}x_{41}x_{23}x_{32}) + (x_{13}x_{22}x_{34}x_{41} - x_{13}x_{34}x_{41}x_{22}) \\
&+ (x_{14}x_{23}x_{41}x_{32} - x_{23}x_{41}x_{32}x_{14})\} \\
= (q-q^{-1})^3\{&(x_{14}x_{23}x_{42}x_{31} - x_{23}x_{14}x_{31}x_{42}) + (x_{13}x_{24}x_{32}x_{41} - x_{24}x_{13}x_{41}x_{32})\} \\
= (q-q^{-1})^4&(x_{14}x_{23}x_{32}x_{41} - x_{23}x_{14}x_{41}x_{32}) = 0
\end{aligned}$$

$$\begin{aligned}
A_0^4 = (q-q^{-1})\{&(x_{24}x_{41}x_{13}x_{32} - x_{32}x_{24}x_{41}x_{13}) + (x_{21}x_{14}x_{43}x_{32} - x_{32}x_{21}x_{14}x_{43}) \\
&+ (x_{21}x_{13}x_{34}x_{42} - x_{34}x_{42}x_{21}x_{13}) + (x_{24}x_{43}x_{31}x_{12} - x_{31}x_{12}x_{24}x_{43}) \\
&+ (x_{23}x_{34}x_{41}x_{12} - x_{34}x_{41}x_{12}x_{23}) + (x_{23}x_{31}x_{14}x_{42} - x_{31}x_{14}x_{42}x_{23}) \\
&+ (x_{24}x_{41}x_{13}x_{32} - x_{32}x_{24}x_{41}x_{13}) + (x_{23}x_{31}x_{14}x_{42} - x_{31}x_{14}x_{42}x_{23})\} \\
= (q-q^{-1})^2\{&(x_{22}x_{31}x_{14}x_{43} - x_{31}x_{22}x_{14}x_{43}) + (x_{34}x_{22}x_{41}x_{13} - x_{34}x_{41}x_{22}x_{13}) \\
&+ (x_{24}x_{31}x_{42}x_{13} - x_{24}x_{31}x_{42}x_{13}) + (x_{21}x_{14}x_{33}x_{42} - x_{21}x_{14}x_{42}x_{33}) \\
&+ (x_{24}x_{33}x_{41}x_{12} - x_{24}x_{41}x_{33}x_{12})\} = 0
\end{aligned}$$

$$\begin{aligned}
A_{-2}^4 = (q-q^{-1})\{&(x_{34}x_{42}x_{21}x_{13} - x_{42}x_{21}x_{13}x_{34}) + (x_{32}x_{24}x_{41}x_{13} - x_{41}x_{13}x_{32}x_{24}) \\
&+ (x_{32}x_{21}x_{14}x_{43} - x_{43}x_{32}x_{21}x_{14}) + (x_{34}x_{41}x_{12}x_{23} - x_{41}x_{12}x_{23}x_{34}) \\
&+ (x_{31}x_{14}x_{42}x_{23} - x_{42}x_{23}x_{31}x_{14}) + (x_{31}x_{12}x_{24}x_{43} - x_{43}x_{31}x_{12}x_{24})\} \\
= (q-q^{-1})^2\{&(x_{33}x_{42}x_{21}x_{14} - x_{42}x_{21}x_{33}x_{14}) + (x_{32}x_{23}x_{41}x_{14} - x_{32}x_{41}x_{23}x_{14}) \\
&+ (x_{32}x_{41}x_{23}x_{14} - x_{41}x_{32}x_{14}x_{23}) + (x_{33}x_{41}x_{12}x_{24} - x_{41}x_{12}x_{33}x_{24}) \\
&+ (x_{31}x_{13}x_{42}x_{24} - x_{42}x_{31}x_{24}x_{13})\} \\
= (q-q^{-1})^3\{&(x_{32}x_{41}x_{24}x_{13} - x_{41}x_{32}x_{13}x_{24}) + (x_{31}x_{42}x_{14}x_{23} - x_{42}x_{31}x_{23}x_{14})\} \\
= (q-q^{-1})^4&(x_{32}x_{41}x_{14}x_{23} - x_{41}x_{32}x_{23}x_{14}) = 0.
\end{aligned}$$

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